ON SETS OF FUNCTIONS OF A GENERAL VARIABLE*

BY

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A theory of functions of a general variable is due to E. H. Moore.† By a general variable is meant a variable of which the range is a class of elements

$$\mathfrak{Q} \equiv [q]$$

entirely unconditioned. Particular instances of the theory may be obtained by specializing the class $\mathfrak{Q}$. For example the class $\mathfrak{Q}$ may consist of a finite number of elements, a denumerable infinitude of elements, or a continuous infinitude of elements. The elements themselves may be numbers, real, complex, or hypercomplex; or they may be without numerical character.

A real (single-valued) function μ on a general range $\mathfrak{Q}$ is a correspondence between the elements of $\mathfrak{Q}$ and a class of real numbers, such that for every element q of $\mathfrak{Q}$ there is a definite corresponding real number, notationally $\mu(q)$.

A property of such a function which is in no way dependent for its definition on any special character which the range $\mathfrak{Q}$ may have in special instances is said to be a *property of general reference*. For example, a function may be: (a) everywhere zero on $\mathfrak{Q}$, or (b) everywhere positive on $\mathfrak{Q}$, or (c) everywhere negative on $\mathfrak{Q}$, or (d) somewhere positive and nowhere negative on $\mathfrak{Q}$, or (e) somewhere negative and nowhere positive on $\mathfrak{Q}$. We shall use the following symbolic statements to indicate that a function μ has these properties respectively:

(a) $\mu = 0\ (\mathfrak{Q})$, (d) $\mu \geqq' 0\ (\mathfrak{Q})$,

(b) $\mu > 0\ (\mathfrak{Q})$, (e) $\mu \leqq' 0\ (\mathfrak{Q})$.

(c) $\mu < 0\ (\mathfrak{Q})$,

In the present paper we shall be particularly interested in functions which have the property (d) or the property (e). A function which has either of these properties will be said to be *M-definite*. In other words an M-definite function is one which is not identically zero and does not change sign on $\mathfrak{Q}$.

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† The theory is called by Professor Moore "General Analysis," and is developed in his New Haven Mathematical Colloquium Lectures, New Haven, 1910.

Properties of general reference may pertain to a *set* of real functions

$$\mu_1, \mu_2, \cdots, \mu_m,$$

each on the general range $\mathfrak{Q}$.* For example, if there exists a set of real constants $c_1, c_2, \cdots, c_m$ (not all zero) such that

$$\sum_{j=1}^{m} c_j\mu_j = 0 \quad (\mathfrak{Q}), \tag{1}$$

the set of functions is said to be linearly dependent; otherwise it is linearly independent. Other properties of general reference for a set of functions are obtained by replacing the sign = in (1) by the signs used in (b)-(e). For the special instance in which $\mathfrak{Q}$ consists of a finite number of elements, these properties have been considered by the author.†

The object of the present paper is to study the condition

$$\sum_{j=1}^{m} c_j\mu_j \geqq' 0 \quad (\mathfrak{Q}), \tag{2}$$

that is, the condition that a given set of functions on a general range admit an M-definite linear combination.

The central feature of the theory is a certain integral-valued function of the set of functions $\{\mu_j\}$ which we shall call the *M-rank* of the set. In terms of it may be stated a necessary and sufficient condition that (2) admit a solution $(c_1, c_2, \cdots, c_m)$ and the maximum number of c's that may be zero in such a solution. These results are stated in §4, the earlier sections being preparatory to the definition of M-rank.

1. **Reduction and composition of a general range $\mathfrak{Q}$ relative to a function on that range.** We consider a class $\mathfrak{Q}$ of elements q, notationally

$$\mathfrak{Q} \equiv [q].$$

Let μ be any real single-valued function on $\mathfrak{Q}$. We shall have occasion to consider three subclasses of $\mathfrak{Q}$, relative to μ, defined as follows:

$$\mathfrak{Q}_{P^{(\mu)}} \equiv [\text{all } q \text{ such that } \mu(q) > 0] \equiv [p^{(\mu)}],$$
$$\mathfrak{Q}_{N^{(\mu)}} \equiv [\text{all } q \text{ such that } \mu(q) < 0] \equiv [n^{(\mu)}],$$
$$\mathfrak{Q}_{Z^{(\mu)}} \equiv [\text{all } q \text{ such that } \mu(q) = 0] \equiv [z^{(\mu)}].$$

* It may be noted, however, that a property of such a *set* of functions can be considered as a property of a *single* function μ' on a composite range $\mathfrak{Q}'$, the elements q' of the range $\mathfrak{Q}'$ being bipartite elements of the form $q' = (q, j)$, the first part q having the range $\mathfrak{Q}$ and the second part j having the finite range consisting of the numbers $1, 2, \cdots, m$.

† Annals of Mathematics, (2), vols. 20, 27, and 28.

From these subclasses we form a certain composite range relative to the function μ,

$$\mathfrak{Q}^{(\mu)} \equiv \mathfrak{Q}_{P^{(\mu)}}\mathfrak{Q}_{N^{(\mu)}} + \mathfrak{Q}_{Z^{(\mu)}},$$

consisting of the logical sum of the two classes

$$\mathfrak{Q}_{P^{(\mu)}}\mathfrak{Q}_{N^{(\mu)}} \equiv [p^{(\mu)}n^{(\mu)}] \text{ and } \mathfrak{Q}_{Z^{(\mu)}} \equiv [z^{(\mu)}].$$

The elements of the new class

$$\mathfrak{Q}^{(\mu)} \equiv [q^{(\mu)}]$$

will therefore be of two kinds: (1) bipartite elements $p^{(\mu)}n^{(\mu)}$ of which the first part $p^{(\mu)}$ ranges over $\mathfrak{Q}_{P^{(\mu)}}$ and the second part $n^{(\mu)}$ ranges over $\mathfrak{Q}_{N^{(\mu)}}$ independently; and (2) unipartite elements $z^{(\mu)}$ ranging over $\mathfrak{Q}_{Z^{(\mu)}}$.

The process here indicated may evidently be repeated. If σ is a real single-valued function on the new range $\mathfrak{Q}^{(\mu)}$, it determines three subclasses of the range, which may be denoted by $\mathfrak{Q}_{P^{(\mu\sigma)}}$, $\mathfrak{Q}_{N^{(\mu\sigma)}}$, and $\mathfrak{Q}_{Z^{(\mu\sigma)}}$; and from these may be formed the composite class

$$\mathfrak{Q}^{(\mu\sigma)} \equiv \mathfrak{Q}_{P^{(\mu\sigma)}}\mathfrak{Q}_{N^{(\mu\sigma)}} + \mathfrak{Q}_{Z^{(\mu\sigma)}}.$$

The process may be repeated indefinitely, provided at each stage a reducing function is available.

It will be noted that if at any stage the reducing function is everywhere positive or everywhere negative, the new composite range will be a null class; while if the reducing function is identically zero, the new composite range will be identical with the old range.

2. **Reduced outer multiplication.** The reducing function μ determines with any second function ν on the range $\mathfrak{Q}$, a real single-valued function on the composite range $\mathfrak{Q}^{(\mu)}$ which we will call their ***reduced outer product,*** and denote by $((\mu\nu))$. It is defined as follows:*

$$((\mu\nu)) \equiv \begin{cases} \mu(p)\nu(n) - \nu(p)\mu(n) & \text{for } pn \text{ on } \mathfrak{Q}_{P^{(\mu)}}\mathfrak{Q}_{N^{(\mu)}}, \\ \nu(z) & \text{for } z \text{ on } \mathfrak{Q}_{Z^{(\mu)}}. \end{cases}$$

This multiplication is not commutative. Its most obvious property is that $((\mu\mu))=0$ on $\mathfrak{Q}^{(\mu)}$. Other properties are developed in the next section.

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* The outer product of two functions $f(x)$ and $g(x)$, where x is a real variable on a closed interval, has been defined as $f(x)g(y)-g(x)f(y)$. See Kowalewski's *Funktionenräume,* Wiener Sitzungsberichte, vol. 120.

3. **Reduction of a set of functions.** Consider a set of real, single-valued functions

$$\{\mu_j\} \qquad \mu_1, \mu_2, \cdots, \mu_m,$$

on a general range $\mathfrak{Q}$.

Relative to any one of the functions, say μ_k, we may determine a second set of m functions

$$\{\mu_j^{(k)}\} \qquad ((\mu_k\mu_1)),\ ((\mu_k\mu_2)),\ \cdots,\ ((\mu_k\mu_m)),$$

each of which is the reduced outer product of the corresponding function in the given set by μ_k. This new set of functions on the composite range $\mathfrak{Q}^{(\mu_k)}$ will be called a reduced set, or more explicitly, the reduction of the set $\{\mu_j\}$ with respect to μ_k. It has the notable property that its kth constituent is identically zero. The usefulness of this type of reduction lies in the following lemma:

If the reducing function μ_k is not M-definite, then the set $\{\mu_j\}$ admits an M-definite linear combination if and only if the same is true of the reduced set $\{\mu_j^{(k)}\}$. More explicitly, every set of constants $c_1, c_2, \cdots, c_m$, satisfying the condition

$$\sum_{j=1}^{m} c_j\mu_j \geqq' 0 \quad (\mathfrak{Q}), \tag{2}$$

will also satisfy the reduced condition

$$\sum_{j=1}^{m} c_j((\mu_k\mu_j)) \geqq' 0 \quad (\mathfrak{Q}^{(\mu_k)})\ ; \tag{3}$$

and conversely, every solution $c_1, c_2, \cdots, c_m$ of (3) *yields a solution of* (2) *if the constant c_k, which is arbitrary in a solution of* (3), *be suitably chosen.*

We note first that if $\mu_k = 0(\mathfrak{Q})$, the proposition is true though trivial, since in that case the two sets $\{\mu_j\}$ and $\{\mu_j^{(k)}\}$ are identically the same. We may then assume in the proof that the function μ_k changes sign on $\mathfrak{Q}$.

Suppose that the condition (2) has a solution $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m$.

Taking account of the three subclasses of the range $\mathfrak{Q}$ relative to the function μ_k, we obtain from this hypothesis the three statements

$$\sum_{j=1}^{m} \bar{c}_j\mu_j(p) \geqq 0 \qquad (p \text{ on } \mathfrak{Q}_P^{(\mu_k)}), \tag{4}$$

$$\sum_{i=1}^{m} \bar{c}_j\mu_j(n) \geqq 0 \qquad (n \text{ on } \mathfrak{Q}_N^{(\mu_k)}), \tag{5}$$

$$\sum_{j=1}^{m} \bar{c}_j\mu_j(z) \geqq 0 \qquad (z \text{ on } \mathfrak{Q}_z^{(\mu_k)}), \tag{6}$$

with the understanding that the sign $\geqq$ has the significance of $\geqq'$ in at least one of the three statements.

Multiplying (4) by $-\mu_k(n)$ and (5) by $\mu_k(p)$ and adding the results, we have

$$\sum_{j=1}^{m} \bar{c}_j[\mu_k(p)\mu_j(n) - \mu_j(p)\mu_k(n)] \geqq 0 \qquad (pn \text{ on } \mathfrak{Q}_P^{(\mu_k)}\mathfrak{Q}_N^{(\mu_k)}).$$

This together with (6) may be written

$$\sum_{j=1}^{m} \bar{c}_j((\mu_k\mu_j)) \geqq' 0 \quad (\mathfrak{Q}^{(\mu_k)}), \tag{7}$$

which proves the first part of the proposition.

Suppose conversely that the condition (3) has a solution $\bar{c}_1, \bar{c}_2, \cdots, \bar{c}_m$, as expressed by (7).

We note first that the constant $\bar{c}_k$ is arbitrary, since its coefficient is identically zero. We may therefore omit the term corresponding to $j=k$ from the summation, indicating the omission by an apostrophe', and write (7) in the two statements

$$\sum_{j=1}^{m}{}' \bar{c}_j[\mu_k(p)\mu_j(n) - \mu_j(p)\mu_k(n)] \geqq 0 \qquad (pn \text{ on } \mathfrak{Q}_P^{(\mu_k)}\mathfrak{Q}_N^{(\mu_k)}), \tag{8}$$

$$\sum_{j=1}^{m}{}' \bar{c}_j\mu_j(z) \geqq 0 \qquad (z \text{ on } \mathfrak{Q}_z^{(\mu_k)}), \tag{9}$$

one of the signs $\geqq$ having the significance of $\geqq'$.

Since $-\mu_k(p)\mu_k(n)$ is positive, we may obtain from (8) an equivalent statement

$$\sum_{j=1}^{m}{}'\bar{c}_j\frac{\mu_j(p)}{\mu_k(p)} \geqq \sum_{j=1}^{m}{}'\bar{c}_j\frac{\mu_j(n)}{\mu_k(n)} \qquad (\mathfrak{Q}_P^{(\mu_k)}\mathfrak{Q}_N^{(\mu_k)}). \tag{10}$$

Now the values on the left side of (10) must have a greatest lower bound, and those on the right a least upper bound, which bounds may or may not coincide. In any case we may choose the arbitrary $\bar{c}_k$ so that

$$\sum_{j=1}^{m}{}'\bar{c}_j\frac{\mu_j(p)}{\mu_k(p)} \geqq -\bar{c}_k \geqq \sum_{j=1}^{m}{}'\bar{c}_j\frac{\mu_j(n)}{\mu_k(n)}.$$

And from this double relation we obtain

$$\sum_{j=1}^{m}{}' \bar{c}_j\mu_j(p) + \bar{c}_k\mu_k(p) \geqq 0, \qquad \sum_{j=1}^{m}{}' \bar{c}_j\mu_j(n) + \bar{c}_k\mu_k(n) \geqq 0,$$

which together with (9) may be written

$$\sum_{j=1}^{m} \bar{c}_j\mu_j \geqq' 0 \quad (\Omega).$$

This completes the proof of the lemma.

The process of reduction may be repeated. As reduction of the set $\{\mu_j\}$ with respect to μ_k yields the set $\{\mu_j^{(k)}\}$, so reduction of this latter set with respect to one of its constituents $\mu_l^{(k)}$ yields a set which we shall denote by $\{\mu_j^{(kl)}\}$.

In general, we define the set

$$\{\mu_j^{(k_1k_2\cdots k_s)}\} \tag{11}$$

as the reduction of the set

$$\{\mu_j^{(k_1k_2\cdots k_{s-1})}\}$$

with respect to the function $\mu_{k_s}^{(k_1k_2\ldots k_{s-1})}$.

The set (11) will be called an sth reduction of the set $\{\mu_j\}$. Clearly there are many sth reductions, depending on the choice of the sequence $k_1, k_2, \cdots, k_s$. If the integers in this sequence are distinct, s constituent functions of the sth reduction are identically zero.

4. **The M-rank of a set of functions.** We recall that a function μ is said to be M-definite if either of the conditions

$$\mu \geqq' 0 \ (\Omega) \text{ or } \mu \leqq' 0 \ (\Omega)$$

is satisfied.

A set of m functions $\{\mu_j\}$ is said to be of M-rank r if at least one of its $(m-r)$th reductions contains an M-definite constituent function while no one of its $(m-r-1)$th reductions contains such a constituent function. If the given set contains an M-definite function the set is of M-rank m. If neither it nor any of its reductions contains such a function it is of M-rank zero.

THEOREM. *A necessary and sufficient condition that the set of m functions admit an M-definite linear combination is that its M-rank be greater than zero.*

If the M-rank is $r(0<r<m)$, then there is a subset of $m-r+1$ of the functions which admits an M-definite linear combination, but there is no subset of $m-r$ functions for which this is true.

First, if the given set $\{\mu_j\}$ admits an M-definite linear combination, its M-rank is greater than zero. For otherwise the $(m-1)$th reduction $\{\mu_j^{(1,2,\cdots,m-1)}\}$ would contain no M-definite function, while the corresponding reduced condition

$$\sum_{j=1}^{m} c_j\mu_j^{(1,2,\cdots,m-1)} \geqq' 0 \qquad (\Omega^{(1,2,\cdots,m-1)})$$

must admit a solution $c_1, c_2, \cdots, c_m$ by the lemma of §3. These two requirements are incompatible since all functions of the $(m-1)$th reduction are zero except one.

Conversely, suppose the M-rank of the given set is $r(>0)$. Then there is an $(m-r)$th reduction of the set which contains an M-definite function. Suppose, for simplicity of notation, it is $\{\mu_j^{(1,2,\cdots,m-r)}\}$, and suppose the $(m-r+1)$th function of this set is M-definite. Then the condition

$$\sum_{j=1}^{m} c_j\mu_j^{(1,2,\cdots,m-r)} \geqq' 0 \qquad (\Omega^{(1,2,\cdots,m-r)})$$

admits a solution $c_1, c_2, \cdots, c_m$, in which

$c_j=0$ for $j>m-r+1$,

c_j is arbitrary for $j<m-r+1$,

$c_{m-r+1}=+1$ or -1 according as $\mu_{m-r+1}^{(1,2,\cdots,m-r)}$ is positive or negative.

Hence by repeated application of the lemma of §3, we find that the condition

$$\sum_{j=1}^{m} c_j\mu_j \geqq' 0 \qquad (\Omega)$$

admits a solution in which $c_j=0(j>m-r+1)$, suitable values being assigned to $c_j(j<m-r+1)$. The given set therefore admits an M-definite linear combination; indeed a subset of $m-r+1$ of them has this property.

It remains to be proved that when the M-rank is r, no subset of $m-r$ of the given functions admits an M-definite linear combination. Suppose for definiteness that the subset

$$\mu_1, \mu_2, \cdots, \mu_{m-r} \qquad (12)$$

did this property. Then by the first proposition of the theorem (already established) the M-rank of this subset must be greater than zero, call it r'. That means that some $(m-r-r')$th reduction of the subset (12) would contain an M-definite constituent function. The corresponding $(m-r-r')$th reduction of the original set of m functions would contain the same M-definite function, and hence the M-rank of the set would be $r+r'$, contrary to our assumption that it was r.

This completes the proof of the theorem.

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